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SOLUTION OF ONE MIXED PROBLEM FOR THE FOURTH ORDER PARTİAL DİFFERENTİAL EQUATİON BY SENSE OF SHİLOV

РЕШЕНИЕ ОДНОЙ СМЕШАННОЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ С ЧАСТНЫМИ ПРОИЗВОДНЫМИ ЧЕТВЕРТОГО ПОРЯДКА ПО ШИЛОВУ

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Abstract: *The mixed problem for the fourth order partial differential equation with Dirichlet boundary conditions is considered. Corresponding spectral problem is constructed. After construction asymptotic of eigenvalues, the important properties of the Green function were studied, the decomposition theorem is proved. As a main result, solution of mixed problem is found in class of parobolic equation by sense Shilov, included wider class of parabolic equations.*

Keywords: *eigenvalues, Green function, fundamental solution, characteristic determinant, spectral problem, the theorem of decomposition, mixed problem.*

The article is dedicated to the solution of a mixed problem with Dirichlet boundary conditions:

$$\frac{\partial u(x,t)}{\partial t} = i\frac{\partial^4 u(x,t)}{\partial x^4} + q(x)\frac{\partial^2 u(x,t)}{\partial x^2}\ ,\ 0<x<1,\ t>0 \tag{1}$$

$$u(x,0)=\varphi(x), \tag{2}$$

$$\begin{gathered} L_1(u)\equiv u(0,t)=0, \\ L_2(u)\equiv u(1,t)=0, \\ L_3(u)\equiv \frac{\partial u(0,t)}{\partial x}=0, \\ L_4(u)\equiv \frac{\partial u(1,t)}{\partial x}=0, \end{gathered} \tag{3}$$

where $q(x)$ and $\varphi(x)$ are complex value functions.

Spectral problem corresponding to the mixed problem (1)-(3) has the form

$$iy^{IV} + q(x)y'' - \lambda^4 y = -\varphi(x), \quad 0<x<1 \tag{4}$$

$$L_k(y)=0, \quad k=\overline{1,4} \tag{5}$$

The roots of the characteristic equation in the Birkhof sense corresponding to equation (4) are as follows:

$$\theta_1=\theta=e^{-\frac{\pi}{8}i},\ \ \theta_2=i\,\theta,\ \ \theta_3=-\theta,\ \ \theta_4=-i\,\theta$$

To find the asymptotic of fundamental solutions the λ-complex plane is divided into eight sectors by the following way [2,4]:

$$S_k=\left\{\lambda:\lambda_1\,tg\frac{3\pi}{8}<(-1)^k\lambda_2<\lambda_1 tg\frac{5\pi}{8}\right\},\ k=1,2\,,$$

$$S_k=\left\{\lambda:\lambda_1\,tg\frac{\pi}{8}<(-1)^k\lambda_2<\lambda_1 tg\frac{3\pi}{8}\right\},\ k=3,4\,,$$

$$S_k = \left\{ \lambda : \lambda_1 \, tg\left(-\frac{\pi}{8}\right) < (-1)^k \lambda_2 < \lambda_1 tg \frac{\pi}{8} \right\},\ k = 5{,}6,$$

$$S_k = \left\{ \lambda : \lambda_1 \, tg\left(-\frac{3\pi}{8}\right) < (-1)^k \lambda_2 < \lambda_1 tg\left(-\frac{\pi}{8}\right) \right\},\ k = 7{,}8.$$

If $q(x) \in C^1[0,1]$, then on each of sectors $S_k \left(k = \overline{1,8}\right)$ at large values of $|\lambda|$ the asymptotics of the fundamental solution of equation (4) has the following representation [6]:

$$\frac{d^m y_n(x,\lambda)}{dx^m} = (\lambda\omega_n)^m \left[1 + \frac{1}{4\lambda\omega_n} \int_0^x q(\tau) d\tau + O\left(\frac{1}{\lambda^2}\right) \right] e^{\lambda\omega_n x} ;$$

$$|\lambda| \to +\infty,\ \lambda \in S_n \left(n = \overline{1,8}\right),\ \ n = \overline{1,4};\ m = \overline{0,3}\ . \tag{6}$$

Green function [2,3] of spectral problem (4), (5) has the form

$$G(x,\xi,\lambda) = \frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}\ ;\ \ \lambda \in S_k,\ k = \overline{1,8}. \tag{7}$$

$\Delta(\lambda)$ is called a characteristic determinant and can be found as follows:

$$\Delta(\lambda) = \begin{vmatrix} L_1(y_1) & L_1(y_2) & L_1(y_3) & L_1(y_4) \\ L_2(y_1) & L_2(y_2) & L_2(y_3) & L_2(y_4) \\ L_3(y_1) & L_3(y_2) & L_3(y_3) & L_3(y_4) \\ L_4(y_1) & L_4(y_2) & L_4(y_3) & L_4(y_4) \end{vmatrix}$$

and auxiliary determinant $\Delta(x,\xi,\lambda)$ is found as follows

$$\Delta(x,\xi,\lambda) = \begin{vmatrix} g(x,\xi,\lambda) & y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) & y_4(x,\lambda) \\ L_1(g)_x & L_1(y_1) & L_1(y_2) & L_1(y_3) & L_1(y_4) \\ L_2(g)_x & L_2(y_1) & L_2(y_2) & L_2(y_3) & L_2(y_4) \\ L_3(g)_x & L_3(y_1) & L_3(y_2) & L_3(y_3) & L_3(y_4) \\ L_4(g)_x & L_4(y_1) & L_4(y_2) & L_4(y_3) & L_4(y_4) \end{vmatrix},$$

where Cauchy function $g(x,\xi,\lambda)$ takes the form [2]

$$g(x,\xi,\lambda) = \pm\frac{1}{2}\sum_{k=1}^{4} z_k(\xi,\lambda) y_k(x,\lambda)$$

"+" if $0 \le \xi \le x \le 1$, "-" if $0 \le x \le \xi \le 1$,

$$z_k(\xi,\lambda) = \frac{V_{4k}(\xi,\lambda)}{V(\xi,\lambda)},\ k = \overline{1,4},$$

$V_{4k}(\xi,\lambda)$ is an algebraic complement of the fourth row element of Vronskian $V(\xi,\lambda)$

To find the asymptotic of eigenvalues of spectral problem (4), (5) the following theorem was used, [1,7]:

Suppose,that $q(x) \in C^1[0,1]$, then the zeros of the characteristic determinant $\Delta(\lambda)$ are countable set, single limit points of which is $\lambda = \infty$ and the following formulas for the asymptotic zeros are true:

$$\mu_n = \frac{2n+1}{2} \cdot \frac{\pi}{\theta},\quad n \to \infty$$

$$\lambda_n^4 = \mu_n^4 - \frac{1}{\theta_4^2}\mu_n^2\int_0^1 q(\tau)d\tau + O(n) \qquad n\to\infty \tag{8}$$

Based on this fact the following theorem of decomposition is proved:

Teorem: Suppose, that functions $q(x)$ and $\varphi(x)$ satisfies to the following conditions $q(x)\in C^1[0,1], \varphi(x)\in C^2[0,1]$, $\varphi(0)=\varphi(1)=\varphi'(0)=\varphi'(1)=0$. Then for the function $\varphi(x)$ following formula of decomposition is true:

$$\varphi(x) = -\frac{1}{2\pi}\sum_k \int_{C_k}\lambda^3\int_0^1 G(x,\xi,\lambda)\varphi(\xi)d\xi\, d\lambda \,,.$$

here C_k-simple contour and contain only one pole of the Green function in the λ-complex plane

Proof:

First let's choose the numbers θ_k $\left(k=\overline{1,4}\right)$ in each sectors S_p $\left(p=\overline{1,8}\right)$ of which the inequalities $\operatorname{Re}\theta_1\lambda \le \operatorname{Re}\theta_2\lambda \le 0 \le \operatorname{Re}\theta_3\lambda \le \operatorname{Re}\theta_4\lambda$, $\lambda\in S_p$ $\left(p=\overline{1,8}\right)$ are satisfies [5]. For that it is enough to make the choice as follows.

$$\begin{array}{lllll}
\theta_1=\omega_3; & \theta_2=\omega_4; & \theta_3=\omega_2; & \theta_4=\omega_1; & \lambda\in S_1;\\
\theta_1=\omega_3; & \theta_2=\omega_2; & \theta_3=\omega_4; & \theta_4=\omega_1; & \lambda\in S_2;\\
\theta_1=\omega_2; & \theta_2=\omega_3; & \theta_3=\omega_1; & \theta_4=\omega_4; & \lambda\in S_3;\\
\theta_1=\omega_2; & \theta_2=\omega_1; & \theta_3=\omega_3; & \theta_4=\omega_4; & \lambda\in S_4;\\
\theta_1=\omega_1; & \theta_2=\omega_2; & \theta_3=\omega_4; & \theta_4=\omega_3; & \lambda\in S_5;\\
\theta_1=\omega_1; & \theta_2=\omega_4; & \theta_3=\omega_2; & \theta_4=\omega_3; & \lambda\in S_6;\\
\theta_1=\omega_4; & \theta_2=\omega_1; & \theta_3=\omega_3; & \theta_4=\omega_2; & \lambda\in S_7;\\
\theta_1=\omega_4; & \theta_2=\omega_3; & \theta_3=\omega_1; & \theta_4=\omega_2; & \lambda\in S_8;
\end{array}$$

Let's multiply the second, third, fourth, fifth column of the determinant $\Delta(x,\xi,\lambda)$ by $\frac{1}{2}z_1(\xi,\lambda)$, $\frac{1}{2}z_2(\xi,\lambda)$, $-\frac{1}{2}z_3(\xi,\lambda)$, $-\frac{1}{2}z_4(\xi,\lambda)$ correspondingly and add to first colomn. After these transformations, let's denote the elements of the first column by $g_0(x,\xi,\lambda)$ and $g_p(\xi,\lambda)$ $\left(p=\overline{1,4}\right)$ and get :

$$g_0(x,\xi,\lambda) = \begin{cases} z_1(\xi,\lambda)y_1(x,\lambda) + z_2(\xi,\lambda)y_2(x,\lambda), & 0\le\xi\le x\le 1\\ -z_3(\xi,\lambda)y_3(x,\lambda) - z_4(\xi,\lambda)y_4(x,\lambda), & 0\le x\le\xi\le 1\end{cases}$$

$$g_{p+1}(\xi,\lambda) = -\sum_{k=1}^{2} z_k(\xi,\lambda)\left.\frac{d^p y_k(x,\lambda)}{dx^p}\right|_{x=1} - \sum_{k=3}^{4} z_k(\xi,\lambda)\left.\frac{d^p y_k(x,\lambda)}{dx^p}\right|_{x=0} \quad, p=\overline{0,3}\ ,$$

here

$$z_k(\xi,\lambda) = \frac{1}{\lambda^3}\frac{V_{4k}}{V}\left[1 - \frac{1}{4\lambda\theta_k}\int_0^\xi q(\tau)d\tau + O\left(\frac{1}{\lambda^2}\right)\right]e^{-\lambda\theta_k\xi} \quad, |\lambda|\to\infty,\ \lambda\in S_p\left(p=\overline{1,8}\right)$$

V - the Vandermonde determinant of the numbers θ_1, θ_2, θ_3, θ_4 , V_{4k} - is algebraic co-factor of the $(4,k)$ element of determinant V.

First, expanding the determinant $\Delta(x,\xi,\lambda)$ in the first row, and then each of the obtained determinants in the first column, we'll get:

$$\frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}=g_0(x,\xi,\lambda)+\sum_{m=1}^{4}\sum_{k=1}^{4}y_m(x,\lambda)g_k(\xi,\lambda)\frac{\Delta_{km}(\lambda)}{\Delta(\lambda)}$$

Let's denote $O_k\ (k=1,2,3,...)$ series of circle with center of the origin in λ-kompleks plane, radiuses of which are increases and satisfies to condition $\lim_{k\to\infty} r_k=+\infty$. Let's choose radiuses of series of circle $O_k\ (k=1,2,3,...)$ under condition, that thare are don't intersect δ-neighborhood of zeroes of the determinant $\Delta(\lambda)$.

$$\int_0^1\frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}\varphi(\xi)d\xi=\sum_{k=1}^{2}y_k(x,\lambda)\int_0^1 z_k(\xi,\lambda)\varphi(\xi)d\xi-\sum_{k=3}^{4}y_k(x,\lambda)\int_0^1 z_k(\xi,\lambda)\varphi(\xi)d\xi+$$

$$+\sum_{m=1}^{4}\sum_{k=1}^{4}y_m(x,\lambda)\int_0^1 g_k(\xi,\lambda)\varphi(\xi)d\xi$$

Integrating by part all integrals of the right side of the previous equality we'll get. $\int_0^1\frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}\varphi(\xi)d\xi=\frac{-i}{\lambda^4}\varphi(x)+\frac{M(x,\xi,\lambda)}{\lambda^5}$,

here function $M(x,\xi,\lambda)$ is bounded on interval $[0,1]$ with respect to x and ξ, and analytical with respect to λ in sectors $\lambda\in S_p\ (p=\overline{1,8})$,

$$\lim_{k\to\infty}\frac{1}{2\pi}\int_{O_k}\lambda^3\int_0^1 G(x,\xi,\lambda)\varphi(\xi)d\xi d\lambda=\sum_k\int_{O_k}\lambda^3\int_0^1 G(x,\xi,\lambda)\varphi(\xi)d\xi$$

Thus, for $\varphi(x)$ we'll get the formula to put in the theorem.

The theorem is proved.

As it is known, that at $\operatorname{Re} q(x)>0,\ 0\le x\le 1$ equation (1) is parabolic in the sense of Shilov [8]. A following theorem allows us to find solution of the mixed problem (1)-(3) not only in case of parabolic in the sense of Shilov, but also wider classes:

Theorem. Suppose, that functions $q(x)$ and $\varphi(x)$ satisfies to the conditions: $q(x)\in C^1[0,1]$, $\varphi(x)\in C^2[0,1]$, $\varphi(0)=\varphi(1)=\varphi'(0)=\varphi'(1)=0$, $\operatorname{Re}\int_0^1 q(\tau)d\tau>0$. Then solution of the mixed problem (1)-(3) is found by the following formula:

$$u(x,t)=-i\sum_{k=1}^{4}\sum_{n=1}^{\infty}\underset{\lambda=\lambda_{km}}{res}\ \lambda^3 e^{\lambda^4 t}\int_0^1 G(x,\xi,\lambda)\varphi(\xi)d\xi\,, \tag{9}$$

here as $\lambda_{kn}\ (k=\overline{1,4};n=1,2,3,...)$ are denoted all zeroes of the Green function of the corresponding spectral problem and asymptotics of the eigenvalues has been found by formula (8).

Solution of the mixed problem is searched as follows

$$u(x,t)=\sum_{k=1}^{4}\sum_{n=1}^{\infty}\underset{\lambda=\lambda_{km}}{res}\ \lambda^3\int_0^1 G(x,\xi,\lambda)z(\xi,t,\lambda)d\xi\,, \tag{10}$$

here $z(\xi,t,\lambda)$ is unknown function. Taking into account (10) in (1) and (2) to find function $z(\xi,t,\lambda)$ we'll get the following Cauchy problem

$$\frac{dz(\xi,t,\lambda)}{dt}=\lambda^4 z(\xi,t,\lambda) \tag{11}$$

$$z(\xi,0,\lambda)=-i\varphi(\xi) \tag{12}$$

Solution of the Cauchy problem (11) - (12) is found by formula

$$z(\xi,t,\lambda) = -i\varphi(\xi)e^{\lambda^4 t}. \qquad (13)$$

Taking into account solution (13) in series (10) we get the folmula (9). It is easy to check that series (9) is the formal solution of the mixed problem (1)-(3).

Using formula (8) the following estimation is true :

$$\left|e^{\lambda_{kn}^4 t}\right| = e^{t\,\mathrm{Re}\,\lambda_{kn}^4 t} = e^{-t\pi^2 n^2 \int_0^1 q(\tau)d\tau + O(n)}, \quad \mathrm{Re}\int_0^1 q(\tau)d\tau > 0$$

It shows, that at $t>0$ and $x\in[0,1]$ the function $u(x,t)$, defined by series (9) and its forth order with respect to x, and first order with respect to t derivatives uniformly and absolutely converges. The theorem is proved.

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Аннотация: *Рассмотрена смешанная задача для дифференциального уравнения с частными производными четвертого порядка с граничными условиями Дирихле. Строится соответствующая спектральная задача. После построения асимптотики собственных значений изучены важные свойства функции Грина, доказана теорема о разложении. Как основной результат, решение смешанной задачи получено в классе пароболических уравнений по Шилову, включающем более широкий класс параболических уравнений.*

Ключевые слова*: собственные значения, функция Грина, фундаментальное решение, характеристический определитель, спектральная задача, теорема разложения, смешанная задача*

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